SHENTON COLLEGE

ATMAS Mathematics Specialist

Test 2 Calculator Free

Name: ....

Time Allowed : 30 minutes

Marks	/30

Materials allowed: No special materials.

Attempt all questions.
All necessary working and reasoning must be shown for full marks.
Where appropriate, answers should be given in exact values.
Marks may not be awarded for untidy or poorly arranged work.

1 If $f(x) = 16 - x^2$ and $g(x) = \sqrt{x}$,

a) Determine the domain and range of the composition $g(f(x))$. (5)

Natural D&R $f(x)$ D: $x \in \mathbb{R}$ R: $y \leq 16$

R $f(x) \rightarrow$ D $g(x)$

=> $g(x)$ input D: $x \leq 16$

but also $x \geq 0$

=> Range $f(x)$ $y \geq 0$

=> Domain $f(x)$ $-4 \leq x \leq 4$.

$g(x)$ D: $0 \leq x \leq 16$

R: $0 \leq y \leq 4$

So for $g(f(x))$, D: $-4 \leq x \leq 4$, R: $0 \leq y \leq 4$.

✓ natural f(x)
✓ link to input for g(x)
✓ natural g(x) condition
✓ work back to f(x)
✓ final D & R.

b) Determine the largest domain for $f(x)$ (which includes $x = -1$) such that $f^{-1}(x)$ exists, and give the equation for $f^{-1}(x)$ on that domain. (2)

Domain $x \leq 0$

Associated inverse is $f^{-1}(x) = -\sqrt{16 - x}$

2 If $f(x) = e^x$ and $g(x) = \frac{1}{x-e}$,

a) Determine $g(f(x))$, giving the domain and range of the composition. (3)

$g(f(x)) = \frac{1}{e^x - e}$

$D: x \neq 1$

$x < 1, \quad R: -\infty < y < -\frac{1}{e}$

$x > 1, \quad R: 0 < y < \infty$

$R: y \in \mathbb{R} / [-\frac{1}{e}, 0]$

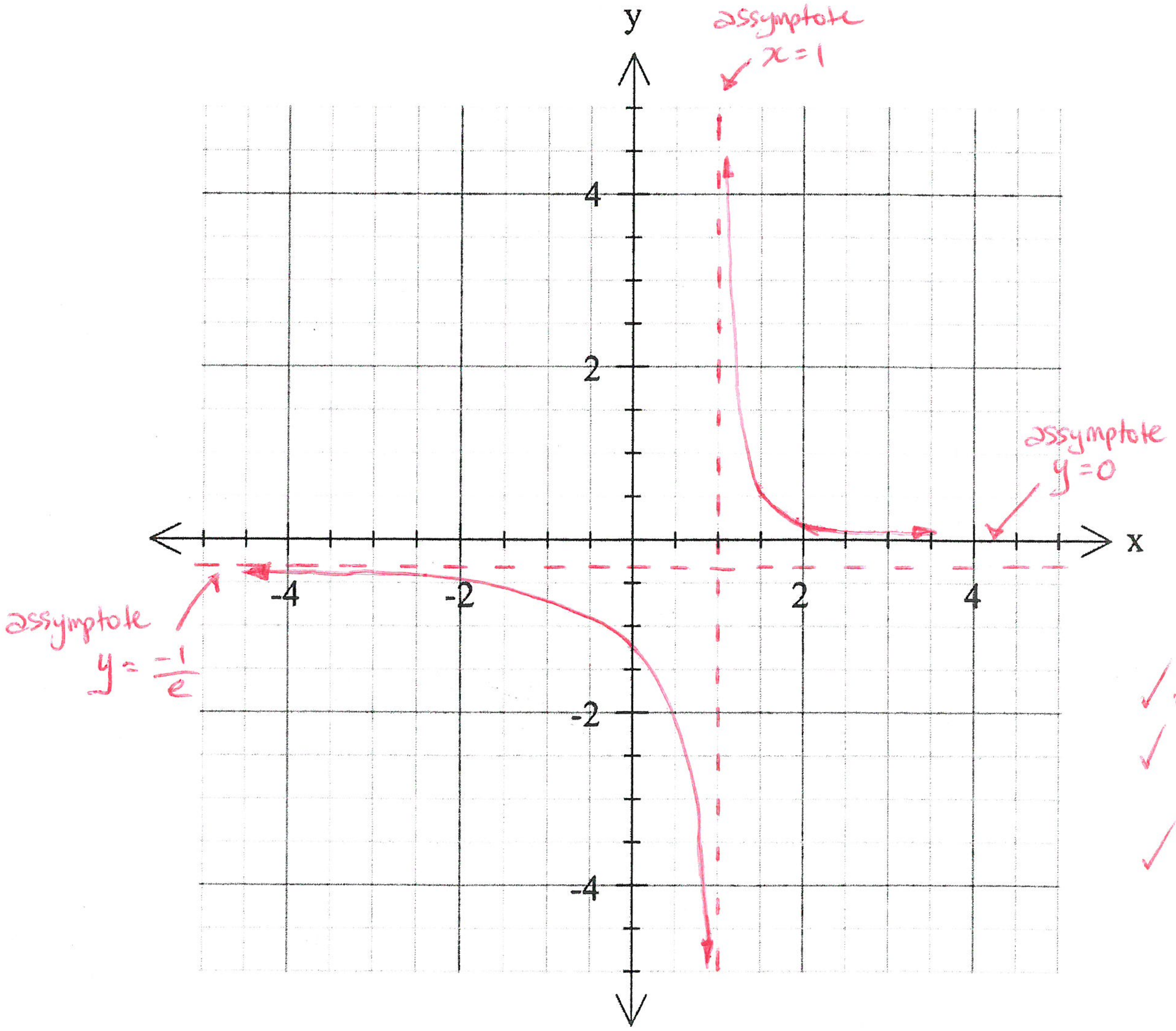
or $y > 0, \; y < -\frac{1}{e}$

✓ composition

✓ exclude $x = 1$ from domain

✓ range exclusion

b) Draw a sketch of the composite function $y = g(f(x))$, indicating any important features. (3)



✓ vertical

✓ both horizontal

✓ curves

3 A function is defined using absolute value notation as $f(x) = |x+3| - |x-4|$

a) Complete the following piecewise definition for the function $f(x)$. (4)

$x < -3, \quad -(x+3) + (x-4) = -7$

$-3 \le x \le 4, \quad (x+3) + (x-4) = 2x-1$

$x > 4, \quad (x+3) - (x-4) = 7$

✓ boundaries at −3 and 4

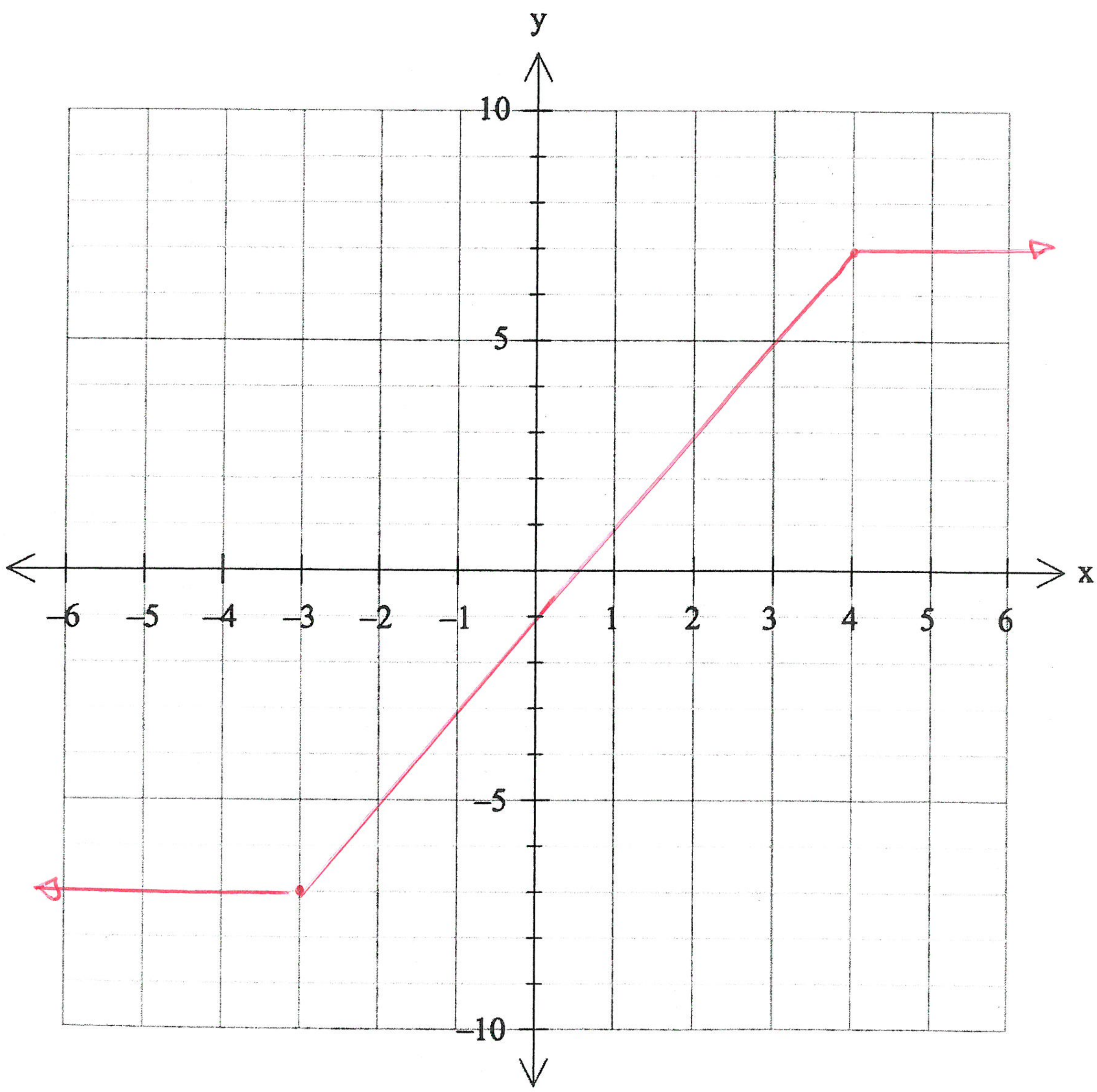
✓ appropriate conversion of | | to ±( )

✓ piecewise functions

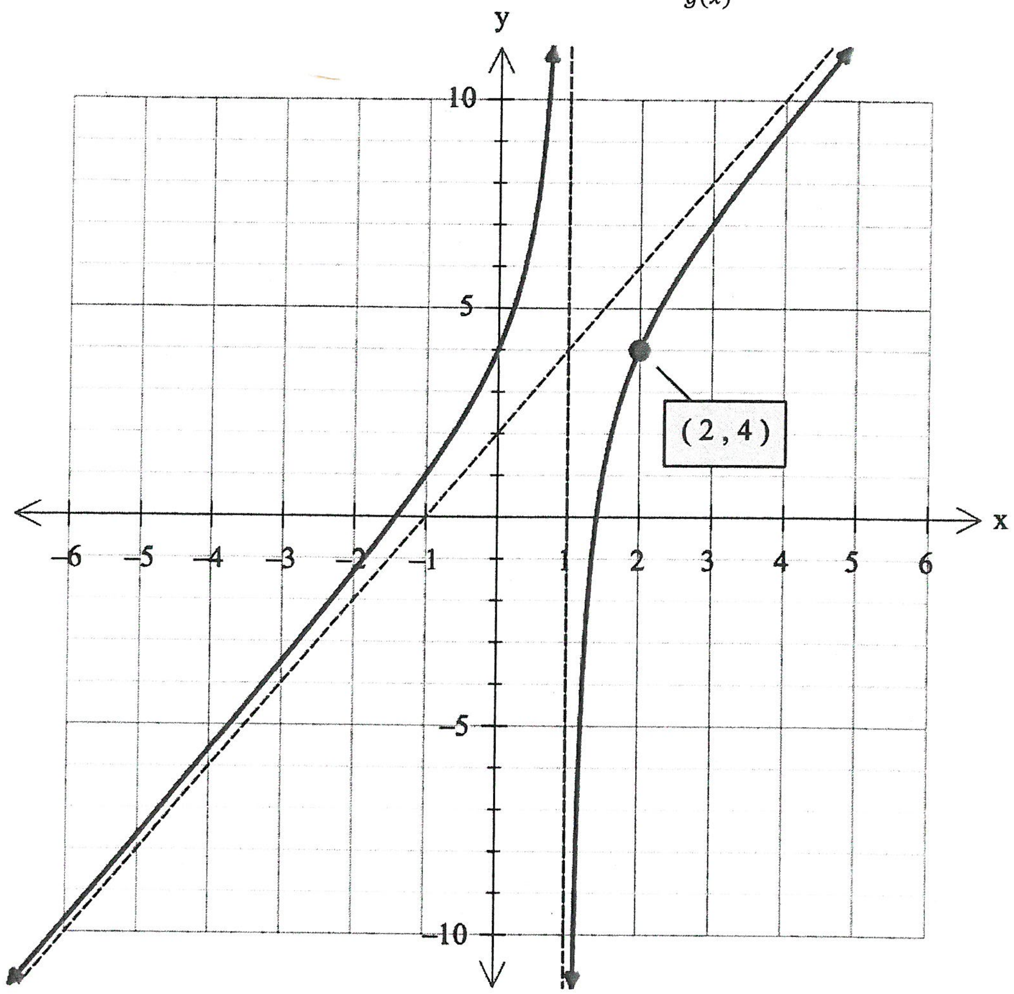
✓ matching domains

$$f(x) = \begin{cases} -7 & \text{for } x < -3 \\ 2x-1 & \text{for } -3 \le x \le 4 \\ 7 & \text{for } x > 4 \end{cases}$$

b) Sketch the function $y = |x+3| - |x-4|$ on the set of axes below. (2)



4 The graph below was created from the function $y = \frac{f(x)}{g(x)}$.



Determine both $f(x)$ and $g(x)$. (3)

Assymptote @ $x = 1$ $\Rightarrow$ $(x-1)$ factor

$\Rightarrow g(x) = x - 1$

✓ identify $g(x)$

Obique assymptote $2x + 2$

$\Rightarrow y = 2x + 2 + \frac{r}{x-1}$

✓ alternative form for y

Using $(2,4)$

$4 = 2(2) + 2 + \frac{r}{1}$

✓ r

$r = -2$

✓ $f(x)$

$y = \frac{(2x+2)(x-1) - 2}{x-1}$

$= \frac{2x^2 - 4}{x-1}$

$\Rightarrow f(x) = 2x^2 - 4$

5 For a line passing through the points $\begin{pmatrix}1\\5\\-2\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\3\end{pmatrix}$, find

a) The vector equation of the line. (2)

$$\tilde{d} = \begin{pmatrix}2\\-1\\3\end{pmatrix} - \begin{pmatrix}1\\5\\-2\end{pmatrix}$$

$$= \begin{pmatrix}1\\-6\\5\end{pmatrix}.$$

Line $\tilde{r} = \begin{pmatrix}1\\5\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\-6\\5\end{pmatrix}$

✓ $\tilde{d}$

✓ equation.

b) The parametric equations of the line. (2)

$$x = 1 + \lambda$$
$$y = 5 - 6\lambda$$
$$z = -2 + 5\lambda$$

c) The point on this line which is closest to the point $\begin{pmatrix}4\\9\\-4\end{pmatrix}$. (4)

Vector from $\begin{pmatrix}4\\9\\-4\end{pmatrix}$ to L, as a function of λ,

$$\begin{pmatrix}1+\lambda\\5-6\lambda\\-2+5\lambda\end{pmatrix} - \begin{pmatrix}4\\9\\-4\end{pmatrix} = \begin{pmatrix}-3+\lambda\\-4-6\lambda\\2+5\lambda\end{pmatrix}$$

Closest point when $\begin{pmatrix}-3+\lambda\\-4-6\lambda\\2+5\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\-6\\5\end{pmatrix} = 0$

$$-3 + \lambda + 24 + 36\lambda + 10 + 25\lambda = 0$$
$$62\lambda = -31$$
$$\lambda = -\tfrac{1}{2}.$$

Point is $\begin{pmatrix}1\\5\\-2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-6\\5\end{pmatrix}$

$$= \begin{pmatrix}\frac{1}{2}\\8\\-\frac{9}{2}\end{pmatrix}.$$

✓ vector from P → L

✓ dot product

✓ λ

✓ point

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ATMAS Mathematics Specialist

Test 2 Calculator Assumed

Name: ....

Time Allowed : 25 minutes

Marks	/30

Materials allowed: Classpad, calculator.

Attempt all questions.
All necessary working and reasoning must be shown for full marks.
Where appropriate, answers should be given to two decimal places.
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1 The position vectors $\overset{A}{\begin{pmatrix}3\\1\\-1\end{pmatrix}}$, $\overset{B}{\begin{pmatrix}-5\\-2\\2\end{pmatrix}}$ and $\overset{C}{\begin{pmatrix}-14\\9\\-2\end{pmatrix}}$ are all points on the plane P_1.

a) Determine the vector equation of P_1. (3)

ALTERNATIVE $\begin{pmatrix}9\\-11\\4\end{pmatrix}$

$\tilde{b} = A - B$
$= \begin{pmatrix}8\\3\\-3\end{pmatrix}$

$\tilde{c} = A - C$
$= \begin{pmatrix}17\\-8\\1\end{pmatrix}$

$\tilde{b} \not\parallel \tilde{c}$

$P_1 = \begin{pmatrix}3\\1\\-1\end{pmatrix} + \alpha\begin{pmatrix}8\\3\\-3\end{pmatrix} + \beta\begin{pmatrix}17\\-8\\1\end{pmatrix}$

✓ $\tilde{b}$
✓ $\tilde{c} \not\parallel \tilde{b}$
✓ P_1

b) Determine the Cartesian equation of P_1. (3)

$\tilde{b} \times \tilde{c} = \begin{pmatrix}-21\\-59\\-115\end{pmatrix}$

$(\tilde{b} \times \tilde{c}) \cdot (A) = -7$
or B or C

$-21x - 59y - 115z = -7$

✓ $\tilde{n}$
✓ $A \cdot \tilde{n}$
✓ cartesian

2 Determine the shortest distance between the parallel planes $3x - 2y + 5z = 7$ and $3x - 2y + 5z = 15$, giving your answer as an exact value. (6)

P_1

$P_1 \quad \tilde{r} \cdot \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 7$

$\Rightarrow \tilde{n} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

point on P_1 is $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ is perpendicular to P_1

Intersecting with P_2 at

$\begin{pmatrix} 3\lambda \\ -1-2\lambda \\ 1+5\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 15$

$9\lambda + 2 + 4\lambda + 5 + 25\lambda = 15$

$\lambda = \frac{-4}{19}$

$\left|\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}\right| = \sqrt{38}$

$\left|\sqrt{38} \times \frac{-4}{19}\right| = \frac{4\sqrt{38}}{19}$

✓ $\tilde{n}$
✓ L
✓ intersects with dot prod form
✓ λ
✓ $|\tilde{n}|$
✓ distance.

3 The equation $4x^2 + y^2 + 8x - 2y - 11 = 0$ describes an ellipse. Determine…

a) The coordinates of the centre. (2)

$4(x+1)^2 - 4 + (y-1)^2 - 1 - 11 = 0$

$4(x+1)^2 + (y-1)^2 = 16$

$\frac{(x+1)^2}{4} + \frac{(y-1)^2}{16} = 1$

centre at $(-1, 1)$

b) The length of the major axis. (1)

8

$(b^2 = 16,\ b = 4,\ 2b = 8)$

c) The length of the minor axis. (1)

4

$(a^2 = 4,\ a = 2,\ 2a = 4)$

d) The domain and range. (2)

$x \in [-3, 1],\ y \in [-3, 5]$

4 Draw a sketch of each of the following rational functions, indicating on your sketch important features such as asymptotes, intercepts, and critical points.

- You may use your Classpad to find intercepts, these do not need to be shown algebraically.
- You may also use your Classpad to calculate any derivates required, however, you must then clearly show how you would interpret the relevant calculus to assist you with your sketch.

a) $y = \dfrac{3x^2}{x-1}$ (6)

$y = 3x + 3 + \dfrac{3}{x-1}$

$\dfrac{dy}{dx} = 3 - \dfrac{3}{(x-1)^2}$

$\dfrac{dy}{dx} = 0 \Rightarrow x = 0$ or $x = 2$
stationary points

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0} < 0$

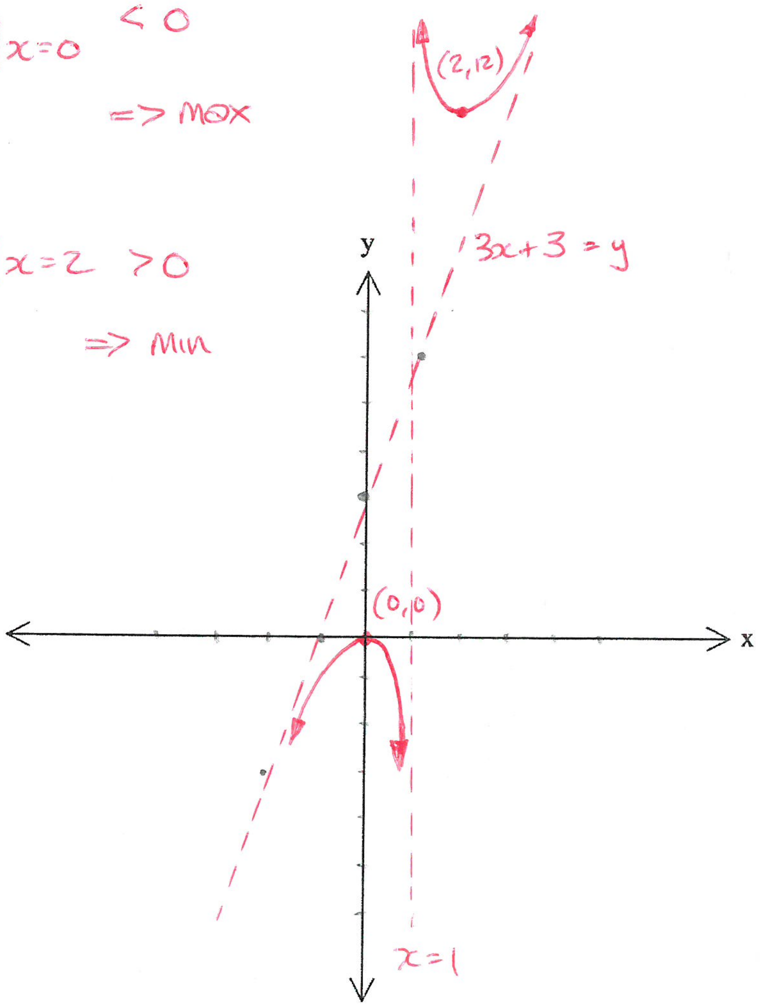
$\Rightarrow$ max

$\left.\dfrac{d^2y}{dx^2}\right|_{x=2} > 0$

$\Rightarrow$ min

✓ $x=1$ assymptote
✓ $3x+3$ 〃
✓ $\dfrac{dy}{dx} = 0$
✓ turning points

Coordinates
$(0,0)$
& $(2,12)$



✓✓ graph

b) $$y = \frac{x^3 - 8}{(x-1)(x+1)}$$ (6)

$$y = \frac{x(x^2-1) + x - 8}{x^2-1}$$

$$= x + \frac{x-8}{x^2-1}$$

$\frac{dy}{dx} = 0 \Rightarrow x = 0$ or $x = -2.91$

stationary points

$\frac{d^2y}{dx^2}\Big|_{x=0} = 16 \Rightarrow$ min

$\frac{d^2y}{dx^2}\Big|_{x=-2.91} = -1.17 \Rightarrow$ max

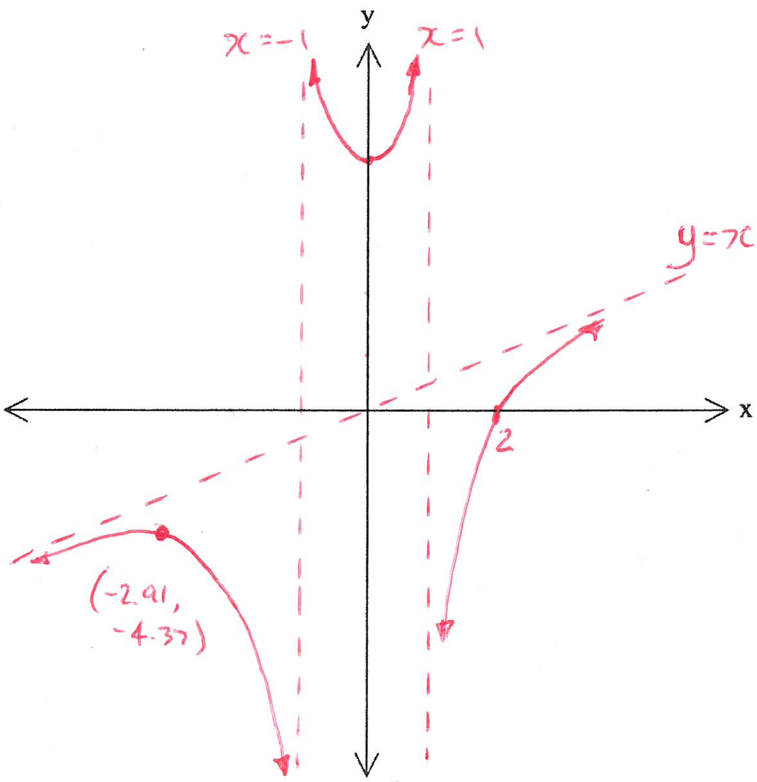
Coordinates $(0, 8)$ & $(-2.91, -4.37)$

✓ $x = 1$ & $x = -1$ assymptotes

✓ $y = x$ assymptote

✓ $\frac{dy}{dx} = 0$

✓ turning points



✓✓ graph.